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On the overlap integral of associated Legendre polynomials

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Abstract. A simple compact closed-form expression is derived from the definite integral $\int_{-1}^{1} P_{\ell_1}^{m_1}(x)P_{\ell_2}^{m_2}(x)\,\mathrm{d}x$ using a convenient analytical formula for the associated Legendre function $P_\ell^m(x)$. This result is easily generalized for integrands involving products of an arbitrary number of associated Legendre polynomials.

1. Introduction

There has been recent interest in deriving closed expressions for the definite integral

$$I(\ell_1, m_1; \ell_2, m_2) = \int_{-1}^{1} P_{\ell_1}^{m_1}(x)P_{\ell_1}^{m_2}(x)\,\mathrm{d}x \tag{1}$$

involving a product of two associated Legendre polynomials. Here, the indices ℓ_1, m_1, ℓ_2, m_2 are integers with $-\ell_1 \leqslant m_1 \leqslant \ell_1$ and $-\ell_2 \leqslant m_2 \leqslant \ell_2$. Salem and Wio [1] developed analytical expressions for (1) with $0 \leqslant m_2 \leqslant \ell_2$ and $m_1 = 0$. This work was extended by Szalay [2] who considered arbitrary non-negative integer values of $m_1$while using $m_2 = m_1 + m$, where m is a positive integer. The results obtained by Salem and Wio [1] and Szalay [2] were, however, rather cumbersome, with expressions differing according to whether $(m_1 + m_2)$ were even or odd integers. Furthermore, the expressions themselves could not be generalized to integrands involving products of an arbitrary number of associated Legendre polynomials.

In this work, a simple closed expression is derived which is applicable regardless of whether or not (m_1+m_2) is an even or odd integer. Furthermore, the expression thus obtained generalizes easily to products of an arbitrary number of associated Legendre polynomials. The paper is organized as follows: section 2 introduces an analytical formula for $P_\ell^m(x)$ which is then used to solve the integral (1). Section 3 summarizes the results obtained.

2. Evaluation of $I(\ell_1, m_1; \ell_2, m_2)$

A closed expression for the associated Legendre polynomial has been available for some years but is not generally well known. Following the approach of Caola [3], it may be shown that the associated Legendre polynomials can be written as

$$P_\ell^m(x) = \sum_{p=0}^{p_{\max}} a_{\ell,m}^p (1-x^2)^{(m+2p)/2} x^{\ell-m-2p} \tag{2}$$

where $\ell = 0, 1, 2, \ldots$ and $0 \leqslant m \leqslant \ell$. The upper limit of the summation is $p_{\max} = [(\ell - m)/2]$, which is the integer part of $(\ell - m)/2$. The coefficient $a^p_{\ell,m}$ is given as

$$a^p_{\ell,m} = \frac{(-1)^p(\ell+m)!}{2^{m+2p}(m+p)!\,p!(\ell-m-2p)!}. \tag{3}$$

Substituting (2) into (1), we obtain

$$I(\ell_1, m_1; \ell_2, m_2) = \sum_{p_1=0}^{p_{1\max}} \sum_{p_2=0}^{p_{2\max}} a^{p_1}_{\ell_1,m_1} a^{p_2}_{\ell_2,m_2} I_{p_1,p_2}(\ell_1, m_1; \ell_2, m_2) \tag{4}$$

where

$$I_{p_1,p_2}(\ell_1, m_1; \ell_2, m_2) = \int_{-1}^{1} \mathrm{d}x (1-x^2)^{(m_1+m_2+2p_1+2p_2)/2} x^{\ell_1+\ell_2-m_1-m_2-2p_1-2p_2}. \tag{5}$$

From (5), we obtain the selection rule

$$I(\ell_1, m_1; \ell_2, m_2) = 0 \tag{6}$$

when $(\ell_1 + \ell_2 - m_1 - m_2)$ is an odd integer, as previously noted by Salem and Wio [1] for $m_1 = 0$. Integral (5) can be easily evaluated by noting that it possesses the form of the beta function. By further expressing the beta function in terms of the gamma functions, and by substituting into (4), we obtain

$$I(\ell_1, m_1; \ell_2, m_2) = \sum_{p_1=0}^{p_{1\max}} \sum_{p_2=0}^{p_{2\max}} a^{p_1}_{\ell_1,m_1} a^{p_2}_{\ell_2,m_2} \times \frac{\Gamma(\frac{1}{2}(\ell_1+\ell_2-m_1-m_2-2p_1-2p_2+1))\Gamma(\frac{1}{2}(m_1+m_2+2p_1+2p_2+2))}{\Gamma(\frac{1}{2}(\ell_1+\ell_2+3))} \tag{7}$$

for $0 \leqslant m_1 \leqslant \ell_1$ and $0 \leqslant m_2 \leqslant \ell_2$, provided that $(\ell_1 + \ell_2 - m_1 - m_2)$ is an even integer. Equation (7) is thus the analytical solution of the integral (1) which involves summing $(p_{1\max} + 1)(p_{2\max} + 1)$ terms of alternating sign.

Equation (7) may be compared with a known simple closed-form expression [4] for the case $m_1 = m_2$. Here, both expressions can be shown to be numerically identical although (7) is more cumbersome in form. However, equation (7) has the advantage of remaining applicable even when $m_1 \neq m_2$.

It is clear that this method can be easily generalized in a straightforward manner for integrands containing a product of n associated Legendre polynomials.

3. Summary

In this paper we have shown that the definite integral of a product of two associated Legendre polynomials can be expressed in terms of a compact formula. This same result is applicable regardless of whether the sum of the m values is an even or an odd integer. Furthermore, this formula has the advantage of being generalized easily to evaluate the definite integral of a product of an arbitrary number of associated Legendre polynomials.

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